

# Reliability Polynomials of Simple Graphs having Arbitrarily many Inflection Points

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## Abstract

In this paper we show that for each  $n$ , there exists a simple graph whose reliability polynomial has at least  $n$  inflection points.

## 1 Introduction

The reliability of a graph  $G$  is the probability that the graph remains connected when each edge is included, or “functions”, with independent probability  $p$ . Equivalently, we can say that each edge fails with probability  $q = 1 - p$ . This function can be written as a polynomial in either  $p$  or  $q$ , though for our purposes it will be convenient to use  $q$ ; for instance, if  $f(q) = R(G)(q)$  for a nontrivial graph  $G$ , then we have  $f(0) = 1$  and  $f(1) = 0$ . Since the first derivative of  $R(G)(q)$  is always negative on  $(0, 1)$ , it is natural to consider whether the second derivative is ever zero, i.e., whether  $R(G)(q)$  has any inflection points.

It is typical for the reliability of a graph to have at least one inflection point, and families of simple graphs with reliability polynomials having two inflection points have been found [2]. In [4], Graves and Milan show that there exist non-simple graphs whose reliability polynomials have at least  $n$  inflection points for any integer  $n$ . They point out that no example is known of a simple graph whose reliability polynomial has more than two inflection points. What we show in this paper is that for each  $n$ , there exists a simple graph whose reliability polynomial has at least  $n$  inflection points.

## 2 Preliminaries

Our proof consists of two major parts: we first demonstrate that there exist reliability polynomials whose second derivative satisfies certain bounds, and then from this collection of polynomials we form products which have arbitrarily many inflection points. The *one-point union* of graphs  $G$  and  $H$ , denoted  $G * H$ , is the graph union where exactly one vertex is chosen from each graph and the chosen vertices are identified. Regardless of the choices made the reliability polynomial of the one-point union of graphs is the product of the reliability

polynomials of each graph. Ultimately, the graphs we use will be one-point unions of complete graphs, and in the following section we demonstrate that the second derivative of such graphs can be made arbitrarily small outside a given interval.

In this section we establish a few facts about reliability polynomials in general. The reliability polynomial of a graph  $G$  can be written in the form

$$R(G)(q) = \sum_{i=0}^m N_i (1-q)^i q^{m-i}, \quad (1)$$

where  $m$  is the number of edges in  $G$ , and  $N_i$  counts the number of connected spanning subgraphs of  $G$  with  $i$  edges.

We first prove a fact about all polynomials having a form similar to (1).

**Proposition 2.1.** *Let  $f(q) = \sum_{i=0}^m N_i (1-q)^i q^{m-i}$ , where the  $N_i$  are either all non-negative or all non-positive. Then for  $q \in [0, 1]$ ,*

$$q(1-q)|f'(q)| \leq m|f(q)|.$$

*Proof.* We first compute

$$f'(q) = \sum_{i=0}^m N_i ((m-i)(1-q)^i q^{m-i-1} - i(1-q)^{i-1} q^{m-i}). \quad (2)$$

Then we have

$$\begin{aligned} q(1-q)|f'(q)| &= \left| \sum_{i=0}^m (m-i)N_i (1-q)^{i+1} q^{m-i} - iN_i (1-q)^i q^{m-i+1} \right| \\ &\leq m(1-q) \left| \sum_{i=0}^m N_i (1-q)^i q^{m-i} \right| + mq \left| \sum_{i=0}^m N_i (1-q)^i q^{m-i} \right| \leq m|f(q)|. \end{aligned}$$

Note that the first inequality, where we bound the coefficients uniformly by  $m$ , uses the hypothesis that the  $N_i$  have the same sign.  $\square$

In the form (1) of a reliability polynomial, the coefficient  $N_i$  counts a subset of the subgraphs of  $G$  with  $i$  edges, and thus we have  $0 \leq N_i \leq \binom{m}{i}$ . Then clearly the above theorem applies when  $f$  is a reliability polynomial. However, we can also consider

$$(1 - R(G))(q) = \sum_{i=0}^m \left( \binom{m}{i} - N_i \right) (1-q)^i q^{m-i}. \quad (3)$$

By the above observation, the coefficients of this polynomial are non-negative, and so Lemma 2.1 applies to  $1 - R(G)$  as well. Next we show that it can also be applied to  $R(G)'(q)$ .

**Proposition 2.2.** *Let  $f(q) = \sum_{i=0}^m N_i (1-q)^i q^{m-i}$  be a reliability polynomial. Then  $f'(q)$  can be written in the same form, and all of the coefficients are non-positive.*

*Proof.* We already computed the derivative of such a function in (2); collecting like terms gives

$$f'(q) = \sum_{i=0}^{m-1} ((m-i)N_i - (i+1)N_{i+1})(1-q)^i q^{m-i-1}. \quad (4)$$

Recall that  $N_i$  represents the number of connected spanning subgraphs of  $G$  with  $i$  edges. Thus we can think of  $(m-i)N_i$  as counting the pairs consisting of a connected spanning subgraph of size  $i$  together with a particular edge not in the subgraph. Similarly,  $(i+1)N_{i+1}$  counts the number of pairs consisting of a connected spanning subgraph of size  $i+1$  and an edge in the subgraph. However, since adding an edge to a connected spanning graph gives another connected spanning graph,  $(m-i)N_i \leq (i+1)N_{i+1}$ . Thus each of the coefficients in the expression for  $f'(q)$  is non-positive.  $\square$

We note that, by the preceding argument, the coefficient of  $(1-q)^i q^{m-i-1}$  in  $R(G)'(q)$  counts the number of pairs consisting of a connected spanning subgraph of size  $i+1$  and a bridge in the subgraph. This gives the following result.

**Proposition 2.3.** *Let  $f(q)$  be the reliability polynomial of a graph  $G$ . If  $G$  has no bridges, then  $f'(0) = 0$ . If  $G$  has at least 3 vertices, then  $f'(1) = 0$ .*

*Proof.* From (4), we see that  $f'(0) = 0$  when the coefficient of  $(1-q)^{m-1}$  is zero. Since there is only a single subgraph of size  $m$ , namely the graph  $G$ , this is equivalent to  $G$  having no bridges.

Similarly,  $f'(1) = 0$  whenever the coefficient of  $q^{m-1}$  is zero. In a subgraph with one edge, that edge is a bridge; thus,  $f'(1) = 0$  if and only if there are no connected spanning subgraphs with one edge, which is clearly true if the graph  $G$  has 3 or more vertices.  $\square$

### 3 Bounding the Reliability of Complete Graphs

For the first half of the proof, we will work directly with the reliability polynomials of complete graphs. We make use of a recurrence relation given in Colbourn's book [3], which we restate here. If  $r_n(q)$  is the reliability of the complete graph  $K_n$ , then

$$r_1 = 1; \\ r_n = 1 - \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} r_k.$$

To make these polynomials easier to work with, we bound them by simpler polynomials.

**Lemma 3.1.** Let  $\alpha = \frac{1}{8}$ , and let  $r_n$  denote  $R(K_n)$ . For  $q \in [0, \alpha]$  and  $n \geq 2$ , we have

$$1 - (n+1)q^{n-1} \leq r_n(q) \leq 1 - (n-1)q^{n-1}.$$

*Proof.* We proceed by induction on  $n$ . The base case where  $n = 2$  is clear, since  $r_2 = 1 - q$ . Now, suppose the claim is true for  $2 \leq k \leq n-1$ . We first note that, since  $r_n = 1 - \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} r_k$ , we can rewrite the inequality we would like to prove as

$$(n-1)q^{n-1} \leq \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} r_k \leq (n+1)q^{n-1}.$$

We will first prove the left hand side of the inequality. From the induction hypothesis,  $r_{n-1} \geq 1 - nq^{n-2}$ , and the fact that  $r_k \geq 0$  for all  $k$ , we have

$$\sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} r_k \geq q^{n-1} + (n-1)q^{n-1}(1 - nq^{n-2}).$$

If we can show that  $q^{n-1}(1 - n(n-1)q^{n-2}) \geq 0$ , it will follow that the right hand side is greater than  $(n-1)q^{n-1}$ , which was what we wanted. If we suppose  $q \leq \frac{1}{6}$ , then it follows that  $n(n-1)q^{n-2} \leq 1$  for  $n \geq 3$ , and the claim holds.

We now proceed to the right hand side of the inequality. Since  $r_k \leq 1$  for all  $k$ , and  $n(n-k) \geq 2n-4$  when  $2 \leq k \leq n-2$ , we have

$$\begin{aligned} \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} r_k &\leq \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} \\ &\leq nq^{n-1} + \sum_{k=2}^{n-2} \binom{n-1}{k-1} q^{k(n-k)} \\ &\leq nq^{n-1} + 2^{n-1} q^{2n-4}. \end{aligned}$$

To show that this is less than or equal to  $(n+1)q^{n-1}$ , it suffices to show that  $2^{n-1}q^{n-3} \leq 1$ . If  $n \geq 4$ , then it is enough to take  $q \leq \frac{1}{8}$ . We may verify directly that

$$r_3 = 1 - q^2 - 2q^2(1 - q) = 1 - 3q^2 + 2q^3 \leq 1 - 2q^2$$

provided  $q \leq \frac{1}{2}$ , which is clearly satisfied since  $q \leq \frac{1}{8}$ .

This completes our induction proof, and so the bounds hold for all  $n \geq 2$ .  $\square$

We also prove another bound which will be used during the proof of the following theorem.

**Lemma 3.2.** Let  $r_n(q) = R(K_n)(q)$ , with  $n \geq 3$ . Then for  $q \in [0, 1]$ ,

$$\frac{r_n(q)}{(1-q)^2} \leq \frac{1}{2} \binom{n}{2}^2.$$

*Proof.* Let  $m = \binom{n}{2}$  denote the number of edges of  $K_n$ . For  $2 \leq i \leq m$ , we have

$$\binom{m}{i} = \frac{m(m-1)}{i(i-1)} \binom{m-2}{i-2} \leq \frac{m^2}{2} \binom{m-2}{i-2}.$$

Recall that we can write  $r_n(q) = \sum_{i=0}^m N_i (1-q)^i q^{m-i}$ . Since  $N_i$  is the number of connected spanning subgraphs of  $K_n$  of size  $i$ , we have  $0 \leq N_i \leq \binom{m}{i}$  for all  $i$ , and  $N_i = 0$  for  $0 \leq i < n-1$ . Since  $n \geq 3$ , we can say

$$\begin{aligned} \frac{r_n(q)}{(1-q)^2} &= \sum_{i=0}^{m-2} N_{i+2} (1-q)^i q^{m-i-2} \leq \sum_{i=0}^{m-2} \binom{m}{i+2} (1-q)^i q^{m-i-2} \\ &\leq \frac{m^2}{2} \sum_{i=0}^{m-2} \binom{m-2}{i} (1-q)^i q^{m-2-i} = \frac{m^2}{2} = \frac{1}{2} \binom{n}{2}^2. \end{aligned} \quad \square$$

Now we have enough in place to prove our first theorem.

**Theorem 3.3.** *Let  $0 < a < b < \frac{1}{8}$ , and let  $\epsilon > 0$ . Then there exists a graph  $G$  such that  $|R(G)''(q)| \leq \epsilon$  when  $q$  in  $[0, 1] \setminus [a, b]$ .*

*Proof.* We first recall that  $r_n$  denotes the reliability of the complete graph  $K_n$  and the inequalities

$$1 - (n+1)q^{n-1} \leq r_n(q) \leq 1 - (n-1)q^{n-1}$$

are valid for  $q \in [0, 1/8]$ . Since we are only interested in upper bounds, we reformulate the left hand side as  $1 - r_n(q) \leq (n+1)q^{n-1}$ .

We compute

$$(r_n^\ell)''(q) = \ell(\ell-1)r_n^{\ell-2}(q)r_n'^2(q) + \ell r_n^{\ell-1}(q)r_n''(q),$$

from which we now obtain various bounds. Note that the coefficients of  $1 - r_n(q)$  are all non-negative. From our lemmas, we have

$$\begin{aligned} q(1-q)|r_n'(q)| &\leq \binom{n}{2}|r_n(q)|; \\ q(1-q)|r_n'(q)| &= q(1-q)|(1-r_n)'(q)| \leq \binom{n}{2}|1-r_n(q)|; \\ q(1-q)|r_n''(q)| &\leq \binom{n}{2}|r_n'(q)|. \end{aligned}$$

For  $q \in [0, 1/8]$ , we have

$$\begin{aligned} q^2(1-q)^2|(r_n^\ell)''(q)| &\leq \ell(\ell-1) \frac{n^2(n-1)^2}{4} |r_n^{\ell-2}(q)| |1-r_n(q)|^2 \\ &\quad + \ell \frac{n^2(n-1)^2}{4} |r_n^{\ell-1}(q)| |1-r_n(q)| \\ &\leq \frac{\ell^2 n^6}{4} (1 - (n-1)q^{n-1})^{\ell-2} q^{2n-2} \\ &\quad + \frac{\ell n^5}{4} (1 - (n-1)q^{n-1})^{\ell-1} q^{n-1}, \end{aligned}$$

where we have, in addition to using the lemmas, simplified  $(n-1)(n+1) \leq n^2$ . We note that  $(1-q)^2 \geq \frac{1}{4}$  when  $q \leq \frac{1}{8}$ , and so we may rearrange this inequality to obtain

$$\begin{aligned} |(r_n^\ell)''(q)| &\leq \ell^2 n^6 (1 - (n-1)q^{n-1})^{\ell-2} q^{2n-4} \\ &\quad + \ell n^5 (1 - (n-1)q^{n-1})^{\ell-1} q^{n-3}. \end{aligned}$$

We denote the two terms in the above bound by  $f(q)$  and  $g(q)$ , respectively. Note that  $f$  and  $g$  are both dependent on our choice of  $\ell$  and  $n$ ; from here on it will be useful to assume that  $n \geq 4$ .

We'd like to demonstrate that  $f(q) + g(q) < \epsilon$  on  $[0, a]$  and  $[b, 1/8]$ . To do this, we first show that  $f + g$  is increasing on the first interval and decreasing on the second; then it suffices to show that  $f(a) + g(a) < \epsilon$  and  $f(b) + g(b) < \epsilon$ .

Taking the derivative of  $f$  and factoring, we see that the sign of  $f'(q)$  is the same as that of the expression

$$-(\ell-2)(n-1)^2 q^{n-1} + (2n-4)(1 - (n-1)q^{n-1}).$$

This expression is non-negative precisely when

$$((\ell-2)(n-1)^2 + (2n-4)(n-1))q^{n-1} \leq 2n-4.$$

After some estimation, we see that if  $\ell n(n-1)q^{n-1} \leq 2(n-2)$ , then  $f'(q) \geq 0$ . Since  $n \geq 4$ , it suffices to show  $\ell n q^{n-1} \leq 1$ .

The sign of  $g'(q)$  is the same as that of

$$-(\ell-1)(n-1)^2 q^{n-1} + (n-3)(1 - (n-1)q^{n-1});$$

similarly, if  $\ell n(n-1)q^{n-1} \leq n-3$ , then  $g'(q) \geq 0$ . Since  $n \geq 4$ , it suffices to show that  $\ell n q^{n-1} \leq \frac{1}{4}$ ; if this is true, then the condition above holds as well, and so  $(f+g)'(q) \geq 0$ . Moreover, since  $\ell n q^{n-1} \leq \ell n a^{n-1}$  for  $q \in [0, a]$ , it is sufficient to show that  $\ell n a^{n-1} \leq \frac{1}{4}$ .

By the same sort of approximation, we see that if  $\ell q^{n-1} \geq 1$ , then  $(f+g)'(q) \leq 0$ . Again, if this is true at  $b$ , then it clearly holds for all  $q \in [b, 1/8]$  as well.

For  $q \in [1/8, 1]$ , we have  $\frac{1}{q^2} \leq 64$ . Using our usual bounds, along with Lemma 3.2, we see

$$\begin{aligned} |(r_n^\ell)''(q)| &\leq \frac{1}{q^2} \ell^2 \binom{n}{2}^2 r_n^{\ell-1}(q) \frac{r_n(q)}{(1-q)^2} \\ &\leq 16 \ell^2 \binom{n}{2}^4 r_n^{\ell-1}(q) \leq \ell^2 n^8 r_n^{\ell-1}(q) \\ &\leq \ell^2 n^8 r_n^{\ell-1}(1/8) \leq \ell^2 n^8 (1 - (n-1)(1/8)^{n-1})^{\ell-1}. \end{aligned}$$

Note that we were able to replace  $q$  with  $1/8$  since  $q \in [1/8, 1]$  and  $r_n$  is decreasing.

We restate our conditions here: we want to find  $\ell \geq 1$  and  $n \geq 4$  such that

- $\ell n a^{n-1} \leq \frac{1}{4}$ ,
- $\ell b^{n-1} \geq 1$ ,
- $f(a) + g(a) \leq \epsilon$ ,
- $f(b) + g(b) \leq \epsilon$ , and
- $\ell^2 n^8 (1 - (n-1)(1/8)^{n-1})^{\ell-1} \leq \epsilon$ .

If we can find  $\ell, n$  satisfying these conditions, then the arguments above show that  $|R(K_n^\ell)''(q)| \leq \epsilon$  for  $q \in [0, 1] \setminus [a, b]$ .

To show that all of the above inequalities can be satisfied, we define a sequence of  $\ell_i$  and  $n_i$  such that the quantities on the left become arbitrarily small (or large, in the case of the second one) for sufficiently large  $i$ . We begin by choosing positive integers  $N, k$  such that  $a < N^{-1/k} < b$ ; this is possible because there is an integer between  $b^{-k}$  and  $a^{-k}$  for sufficiently large  $k$ . Then, we let  $\ell_i = N^i$  and  $n_i = ik$ .

Consider the first expression: we rewrite

$$\ell_i n_i a^{n_i-1} = \frac{ik}{a} (Na^k)^i,$$

which becomes small as  $i$  goes to infinity, by an application of l'Hospital's rule and the fact that  $Na^k < 1$ . Similarly, we can write

$$\ell_i b^{n_i-1} \geq (Nb^k)^i,$$

which tends to infinity since  $Nb^k > 1$ .

For the next two inequalities, we analyze  $f$  and  $g$  separately. First, we have

$$\begin{aligned} \log(f(q)) &= 2 \log(\ell_i) + 6 \log(n_i) \\ &\quad + (\ell_i - 2) \log(1 - (n_i - 1)q^{n_i-1}) + (2n_i - 4) \log(q). \end{aligned}$$

Note that  $\log(1 - (n_i - 1)q^{n_i-1}) \leq -(n_i - 1)q^{n_i-1}$ ; since we'd like to show that  $\log(f(q)) \rightarrow -\infty$  as  $i \rightarrow \infty$ , we may make this replacement. Thus we must show that the expression

$$2i \log(N) + 6 \log(ik) - (N^i - 2)(ik - 1)q^{ik-1} + 2i \log(q^k) - 4 \log(q)$$

becomes arbitrarily small as  $i$  tends to infinity. The term  $4 \log(q)$  is constant with respect to  $i$ , and so after some rearrangement, we need only consider

$$\begin{aligned} &2i \log(N) + 6 \log(ik) - (N^i - 2)(ik - 1)q^{ik-1} + 2i \log(q^k) \\ &\leq 2i \log(Nq^k) + 6 \log(ik) - N^{-1}(Nq^k)^i \end{aligned}$$

For  $q = a$ , the first term tends to negative infinity, and dominates the second, while the third term is also negative; thus  $\log(f(a))$  can be made arbitrarily small, and  $f(a) < \frac{\epsilon}{2}$  for sufficiently large  $i$ . For  $q = b$ , we have  $Nq^k > 1$ , and

so the third term dominates the first two terms, and again we can make  $f(b)$  arbitrarily small.

Proceeding similarly, we collect the nonconstant terms of  $\log(g(q))$ , and make the same upward approximation as before :

$$\begin{aligned} & \log(\ell) + 5 \log(n) - (\ell - 1)(n - 1)q^{n-1} + n \log(q) \\ &= i \log(Nq^k) + 5 \log(ik) - (N^i - 1)(ik - 1)q^{ik-1} \\ &\leq i \log(Nq^k) + 5 \log(ik) - N^{-1}(Nq^k)^i. \end{aligned}$$

The arguments showing that  $g(a)$  and  $g(b)$  become arbitrarily small for sufficiently large  $i$  are identical to those given above for  $f$ .

Finally, we consider the expression in the fifth inequality. We have

$$\begin{aligned} & \log(\ell^2 n^8 (1 - (n - 1)(\frac{1}{8})^{n-1})^{\ell-1}) \\ &\leq 2i \log(N) + 8 \log(ik) - (N^i - 1)(ik - 1)(\frac{1}{8})^{ik-1} \\ &\leq 2i \log(N) + 8 \log(ik) - N^{-1}(N(\frac{1}{8})^k)^i. \end{aligned}$$

Since  $N(\frac{1}{8})^k > 1$ , the last term dominates the others, and evidently this can also be made arbitrarily small.

Thus, if we take  $i$  sufficiently large, we see that  $G = K_{n_i}^{\ell_i}$  satisfies the desired condition.  $\square$

## 4 Main Result

Now we proceed to the proof that there are reliability polynomials with arbitrarily many inflection points. We choose a collection of intervals

$$I_{k,m} = (a_{k,m}, b_{k,m}) \subset [0, 1],$$

for  $k \geq 0$  and  $m \geq 1$ , such that

- $0 < a_{0,1} < b_{0,1} < a_{1,1} < b_{1,1} < a_{2,1} < \dots$ , and  $b_{k,1} < \frac{1}{8}$  for all  $k \geq 0$ ;
- $I_{k,m+1} \subset I_{k,m}$ ;
- $\ell(I_{k,m}) = b_{k,m} - a_{k,m} \leq 2^{-m}$ .

For  $n \geq 3$ ,  $m \geq 1$ ,  $K_n^m$  has at least three vertices and no bridges. By our previous theorem, along with Lemma 2.3, we can find a collection of reliability polynomials  $s_{k,m} : [0, 1] \rightarrow [0, 1]$ , satisfying the following properties.

- $s_{k,m}(0) = 1$  and  $s_{k,m}(1) = 0$ ;
- $s'_{k,m}(0) = s'_{k,m}(1) = 0$ , and  $s'_{k,m}(q) \leq 0$  for all  $q$ ;
- $|s''_{k,m}(q)| \leq 2^{-m-1}$  for  $q \notin I_{k,m}$ .



We now collect the consequences as a series of lemmas.

**Lemma 4.1.** *If  $q \leq a_{k,m}$ , then  $|s'_{k,m}(q)| \leq 2^{-m-1}$  and  $|1 - s_{k,m}(q)| \leq 2^{-m-1}$ . If  $q \geq b_{k,m}$ , then  $|s'_{k,m}(q)| \leq 2^{-m-1}$  and  $|s_{k,m}(q)| \leq 2^{-m-1}$ .*

*Proof.* Consider the first statement; we suppose otherwise and apply the mean value theorem. That is, we suppose there is a  $c \in [0, a_{k,m}]$  such that  $|s'_{k,m}(c)| \geq 2^{-m-1}$ . But this implies that there is a  $d \in [0, c]$  with

$$|s''_{k,m}(d)| = \frac{|s'_{k,m}(c) - s'_{k,m}(0)|}{|c - 0|} > 2^{-m-1},$$

contradicting our assumption about the  $s_{k,m}$ .

Similarly applying the mean value theorem again shows that if there was a  $c \in [0, a_{k,m}]$  with  $|s_{k,m}(c) - 1| \geq 2^{-m-1}$  there would be a  $d$  at which the above bound is not satisfied, another contradiction.

The arguments for  $q \in [b_{k,m}, 1]$  are similar.  $\square$

Now we'd like to show how *large* the derivatives must be on the intervals  $I_{k,m}$ . In particular, we'd like to find a single point in each interval at which both the first and second derivatives are "sufficiently large".

**Lemma 4.2.** *In each  $I_{k,m}$ , there is a point  $q_{k,m}$  satisfying  $s'_{k,m}(q_{k,m}) < -2^{m-2}$  and  $s''_{k,m}(q_{k,m}) > 2^{2m-2}$ .*

*Proof.* We note that for every  $k \geq 0$ ,  $m \geq 1$ , we have  $s_{k,m}(a_{k,m}) \geq \frac{3}{4}$  and  $s_{k,m}(b_{k,m}) \leq \frac{1}{4}$ ; in particular, the difference is at least  $\frac{1}{2}$ . Then by the mean value theorem, there is a  $c \in I_{k,m}$  where

$$|s'_{k,m}(c)| = \frac{|s_{k,m}(b_{k,m}) - s_{k,m}(a_{k,m})|}{|b_{k,m} - a_{k,m}|} \geq \frac{2^{-1}}{2^{-m}} = 2^{m-1}.$$

Note that since  $s'_{k,m}(q) \leq 0$  for all  $q$ , we have  $s'_{k,m}(c) = -2^{m-1}$ . Now let  $d$  be the smallest number such that  $d > c$  and  $s'_{k,m}(d) = -2^{m-2}$ ; applying the mean value theorem again shows that there is a  $q_{k,m} \in [c, d] \subset I_{k,m}$  satisfying

$$s''_{k,m}(q_{k,m}) = \frac{s'_{k,m}(d) - s'_{k,m}(c)}{d - c} \geq \frac{2^{m-2}}{2^{-m}} = 2^{2m-2}.$$

Note that by the definition of  $d$ , we must have  $s'_{k,m}(q_{k,m}) \leq s'_{k,m}(d) = -2^{m-2}$ .  $\square$

Now we show that there exist products of  $s_{k,m}$  (that is, reliability functions of one point unions of complete graphs) with arbitrarily many inflection points. This proof is modeled after the proof of the main result given in [4].

**Theorem 4.3.** *There exist functions  $g_k : [0, 1] \rightarrow [0, 1]$  for  $k \geq 0$ , such that  $g_k = s_{k,m_k} g_{k-1}$  for  $k \geq 1$ , and  $g_k$  has at least  $2k$  inflection points.*

*Proof.* We prove this by induction on  $k$ . We are going to show that there is a sequence of reliability polynomials  $g_k$ , integers  $m_k$ , and points  $q_k \in I_{k,m_k}$ , for  $k \geq 0$ , such that

- $g_k''(q_i) > 0$  for  $i = 0, \dots, k$ ;
- $\int_{q_{i-1}}^{q_i} g_k''(q) dq < 0$  for  $i = 1, \dots, k$ ;
- $g_k = s_{k,m_k} g_{k-1}$  if  $k \geq 1$ .

We begin with the base case  $k = 0$ . We simply let  $g_0(q) = s_{0,1}$ ,  $q_0 = q_{0,1}$ . By our lemma,  $s_{0,1}''(q_{0,1}) > 0$ , and so the first condition holds; the other conditions hold vacuously.

Now suppose that we have found a  $g_{k-1}$  satisfying the above properties, and we would like to find an  $m_k$  such that  $g_k = s_{k,m_k} g_{k-1}$  also satisfies them.

We first consider

$$(s_{k,m} g_{k-1})' = s_{k,m}' g_{k-1} + s_{k,m} g_{k-1}'. \quad (5)$$

For each  $i < k$ ,  $q_i < a_{k,1}$ , so  $\lim_{m \rightarrow \infty} s_{k,m}'(q_i) = 0$  and  $\lim_{m \rightarrow \infty} s_{k,m}(q_i) = 1$ . It follows that

$$\lim_{m \rightarrow \infty} (s_{k,m} g_{k-1})'(q_i) = g_{k-1}'(q_i).$$

Since

$$\int_{q_{i-1}}^{q_i} (s_{k,m} g_{k-1})''(q) dq = (s_{k,m} g_{k-1})'(q_i) - (s_{k,m} g_{k-1})'(q_{i-1}),$$

we have

$$\lim_{m \rightarrow \infty} \int_{q_{i-1}}^{q_i} (s_{k,m} g_{k-1})''(q) dq = \int_{q_{i-1}}^{q_i} g_{k-1}''(q) dq < 0.$$

Now we look at  $(s_{k,m} g_{k-1})'(q_{k,m})$ , noting that the precise location of  $q_{k,m}$  depends on  $m$ . By construction, we have  $s_{k,m}'(q_{k,m}) \leq -2^{m-2}$ , but  $|s_{k,m}(q_{k,m})| \leq 1$ ; thus, by taking  $m$  sufficiently large, we can guarantee that  $(s_{k,m} g_{k-1})'(q_{k,m}) - (s_{k,m} g_{k-1})'(q_{k-1}) < 0$ .

Next we need to consider

$$(s_{k,m} g_{k-1})'' = s_{k,m}'' g_{k-1} + 2s_{k,m}' g_{k-1}' + s_{k,m} g_{k-1}''. \quad (6)$$

For  $q < a_{k,1}$ , we have  $\lim_{m \rightarrow \infty} (s_{k,m} g_{k-1})''(q) = g_{k-1}''(q)$ , since  $\lim_{m \rightarrow \infty} s_{k,m}''(q) = 0$ ,  $\lim_{m \rightarrow \infty} s_{k,m}'(q) = 0$ , and  $\lim_{m \rightarrow \infty} s_{k,m}(q) = 1$ . Thus by the induction hypothesis, for sufficiently large  $m$  we have  $(s_{k,m} g_{k-1})''(q_i) > 0$  for  $i = 0, \dots, k-1$ .

Now we need only consider  $(s_{k,m} g_{k-1})''(q_{k,m})$ . Since  $g_{k-1}$  is a reliability polynomial,  $g_{k-1}' \leq 0$ , so the second term in the expansion of  $(s_{k,m} g_{k-1})''$  is positive. Since  $s_{k,m} < 1$ , the third term is bounded by the maximum of  $|g_{k-1}''|$  on  $[0,1]$ . Finally, since  $g_{k-1}$  is bounded away from 0 on  $I_{k,1}$  and  $\lim_{m \rightarrow \infty} s_{k,m}''(q_{k,m}) = \infty$ , it follows that  $(s_{k,m} g_{k-1})''(q_{k,m}) > 0$  for sufficiently large  $m$ .

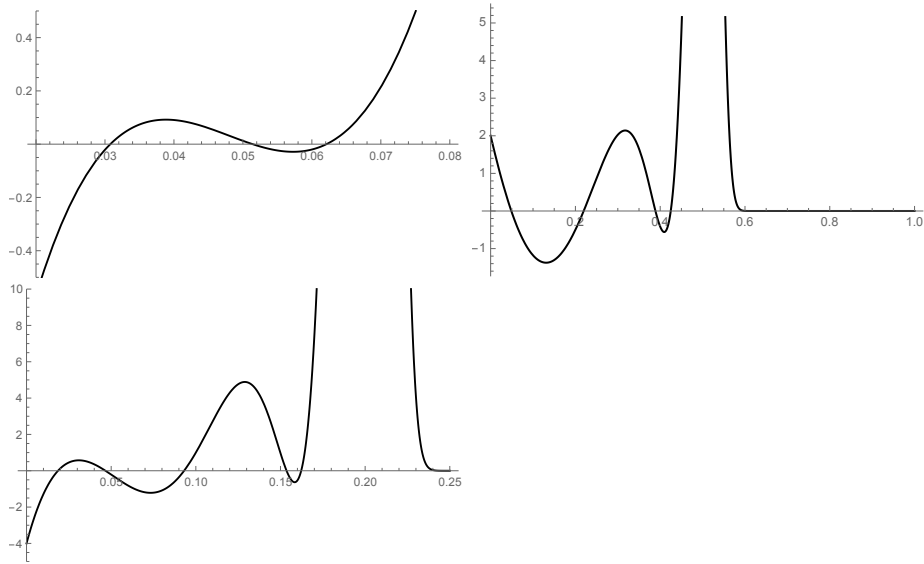


Figure 1: The graphs of the second derivatives of  $R(K_2^5 * K_3^4 * K_4^3 * K_5^{92})$ ,  $R(K_2^2 * K_4^2 * K_{14}^{750})$ , and  $R(K_2^5 * K_3^4 * K_5^{116} * K_{14}^{100,000,000})$  respectively.

If we let  $m_k$  be sufficiently large such that all of the above constructions hold, and define  $q_k = q_{k,m_k}$  and  $g_k = s_{k,m_k} g_{k-1}$ , then  $g_k$  satisfies the induction hypothesis; thus we have constructed the desired  $g_k$  for all  $k \geq 0$ . We know that  $g_k''(q_{i-1}) > 0$  and  $g_k''(q_i) > 0$  for each  $i = 1, \dots, k$ ; but  $\int_{q_{i-1}}^{q_i} g_k'' dq < 0$  tells us that  $g_k''$  is negative somewhere on  $[q_{i-1}, q_i]$ , which implies that there are at least 2 inflection points on this interval. Thus  $g_k$  has at least  $2k$  inflection points, which was what we wanted to show.  $\square$

Note that the polynomial  $g_k$  constructed in the above theorem is the product of the reliability polynomials of a number of complete graphs, and thus it is the reliability polynomial of the one-point union of those graphs. In Figure 1, the second derivatives of three such graphs are shown, demonstrating reliability polynomials of simple graphs having 3, 4, and 5 inflection points.

## 5 Acknowledgements

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